Stabilization of Chaotic Systems with Phase Coupling

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We study phase coupled chaotic systems. Using the Kronecker product we derive a recurrence relation for calculating the Ljapunov exponents of the phase coupled system. We investigate whether phase coupling can be used to control dynamical systems with chaotic behaviour.

1. Introduction

Controlling nonlinear dynamical systems with chaotic behaviour is of critical importance in science and engineering. The most used methods to control unstable periodic orbits are the Ott-Grebogi-Yorke (OGY) [1], least squares and linear least squares methods [2]. We first give a short summary of these methods. Then we describe the phase coupling method to control chaotic behaviour.

Assume the system $\boldsymbol{x}_{t+1} = \boldsymbol{F}(\boldsymbol{x}_t)$ has a periodic orbit $\boldsymbol{x}_{P,i} \in \mathbf{R}^n$, $i = 1, 2, \ldots, P$. We assume that $\boldsymbol{F}$ is an analytic function. Introducing the parameters $\boldsymbol{p}_t \in \mathbf{R}^k$ for the t-th iteration:

$$\boldsymbol{x}_{t+1} = \boldsymbol{F}(\boldsymbol{x}_t, \boldsymbol{p}_t),\ t = 0, 1, 2, \ldots \tag{1}$$

and the deviation

$$\Delta\boldsymbol{x}_t := \boldsymbol{x}_t - \boldsymbol{x}_{P,i}, \tag{2}$$

where $\boldsymbol{x}_t$ lies in the neighbourhood of $\boldsymbol{x}_{P,i}$, $\boldsymbol{x}_{t+1}$ lies in the neighbourhood of $\boldsymbol{x}_{P,i+1}$ and $\boldsymbol{x}_{P,i+1} = \boldsymbol{F}(\boldsymbol{x}_{P,i}, \mathbf{o})$, gives

$$\Delta\boldsymbol{x}_{t+1} = \boldsymbol{F}(\boldsymbol{x}_t, \boldsymbol{p}_t) - \boldsymbol{x}_{P,i+1}. \tag{3}$$

A minimization condition on $\Delta\boldsymbol{x}_{t+1}$ gives the explicit value of $\boldsymbol{p}_t$. The Ott-Grebogi-Yorke (OGY) method follows from linearization of the dynamics of $\boldsymbol{x}_{t+1} = \boldsymbol{F}(\boldsymbol{x}_t, \boldsymbol{p}_t)$ around the periodic points $\boldsymbol{x}_{P,i}$. For the case with one parameter p_t we have

$$\begin{aligned} \Delta\boldsymbol{x}_{t+1} &= \boldsymbol{F}(\boldsymbol{x}_j, p_t) - \boldsymbol{x}_{P,i+1} && (4)\\ &= M_i \Delta\boldsymbol{x}_t + \boldsymbol{h}_i p_t + \ldots, && (5) \end{aligned}$$

where

$$M_i := \left.\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}_{P,i}, p_t=0}, \quad \boldsymbol{h}_i := \left.\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}p_t}\right|_{\boldsymbol{x}=\boldsymbol{x}_{P,i}, p_t=0}. \tag{6}$$

p_t is determined by the condition that $\boldsymbol{x}_{t+1}$ lies along the stable direction of $\boldsymbol{x}_{P,i+1}$. The least squares minimization condition is given by

$$\frac{\partial}{\partial p_i} \|\Delta\boldsymbol{x}_{t+1}(\boldsymbol{p})\|^2 = 0\ i = 1, 2, \ldots, k, \tag{7}$$

where $\boldsymbol{p} = (p_1, p_2, \ldots, p_k)^{\mathrm{T}} \in \mathbf{R}^k$ and $\|\cdot\|$ denotes a norm. Applying this condition to the linearized dynamics of $\boldsymbol{x}_{t+1}$ gives the linear least squares control method.

2. Control in a Phase Coupled System

Consider the discrete dynamical system with $\boldsymbol{F} : \mathbf{R}^m \to \mathbf{R}^m$

$$\boldsymbol{x}_{t+1} = \boldsymbol{F}(\boldsymbol{x}_t) \tag{8}$$

with a given initial value $\boldsymbol{x}_0$ and $\boldsymbol{F}$ an analytic function. The phase coupled system is [3, 4, 5]

$$\begin{aligned} \boldsymbol{x}_{t+1} &= \boldsymbol{F}(\boldsymbol{x}_t) + c(\boldsymbol{u}_t - \boldsymbol{x}_t), && (9)\\ \boldsymbol{u}_{t+1} &= \boldsymbol{F}(\boldsymbol{u}_t) + c(\boldsymbol{x}_t - \boldsymbol{u}_t). && (10) \end{aligned}$$

We define the phase difference

$$\Theta_{t+1} := \boldsymbol{x}_{t+1} - \boldsymbol{u}_{t+1}. \tag{11}$$

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Then

$$\Theta_{t+1} = \boldsymbol{F}(\boldsymbol{x}_t) - \boldsymbol{F}(\boldsymbol{u}_t) - 2c\Theta_t. \quad (12)$$

Using a Taylor expansion for $\boldsymbol{F}(\boldsymbol{x})$ with respect to $\boldsymbol{x}$ and $\boldsymbol{F}(\boldsymbol{u})$ with respect to $\boldsymbol{u}$ and the fact that

$$\frac{\partial \boldsymbol{F}(\boldsymbol{x}=\boldsymbol{x}_t)}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{F}(\boldsymbol{u}=\boldsymbol{x}_t)}{\partial \boldsymbol{u}},$$

we find

$$\Theta_{t+1} = (A_t - 2cI_m)\Theta_t + O(\Theta_t^2), \quad (13)$$

where $A_t := \partial \boldsymbol{F}(\boldsymbol{x}=\boldsymbol{x}_t)/\partial \boldsymbol{x}$ and $O(\Theta^2)$ indicates higher order terms in Θ. Neglecting higher order terms, we find

$$\Theta_t = (A_{t-1} - 2cI_m)(A_{t-2} - 2cI_m) \ldots (A_0 - 2cI_m)\Theta_0, \quad (14)$$

where I_m is the $m \times m$ unit matrix. Equation (14) can also be written in the form

$$\Theta_t = \Big[\sum_{i=0}^{t} \Big((-2c)^{t-i} S_{i,t}\Big)\Big]\Theta_0, \quad (15)$$

$$S_{0,t} := I_m, \ t \in \mathbf{N}_0, \quad (16)$$

$$S_{i,t} := \sum_{0 \le j_1 < j_2 < \ldots < j_i \le t-1} A_{j_i} A_{j_{i-1}} \ldots A_{j_1}, \quad (17)$$

$$i = 1, 2, \ldots, t.$$

Let

$$D_t := (A_t - cI_m) \otimes B_1 + (U_t - cI_m) \otimes B_2 + cI_m \otimes (B_3 + B_4) \quad (18)$$

$$D^t := D_{t-1} D_{t-2} \ldots D_0, \quad (19)$$

where

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, B_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$B_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, U_t := \frac{\partial \boldsymbol{F}(\boldsymbol{x}=\boldsymbol{u}_t)}{\partial \boldsymbol{x}}$$

and $\otimes$ denotes the Kronecker product [6]. Introducing

$$P_t := (A_t - cI_m) \text{ and } Q_t := (U_t - cI_m)$$

we find

$$D^t = F_{t-1}(P,Q) \otimes B_1 + F_{t-1}(Q,P) \otimes B_2 + G_{t-1}(P,Q) \otimes B_3 + G_{t-1}(Q,P) \otimes B_4, \quad (20)$$

$$F_{t+1}(P,Q) = P_{t+1} F_t(P,Q) + cG_t(Q,P), \quad (21)$$

$$F_0(P,Q) = P_0,$$

$$G_{t+1}(P,Q) = P_{t+1} G_t(P,Q) + cF_t(Q,P), \quad (22)$$

$$G_0(P,Q) = cI_m.$$

Rewriting (21) and (22) as

$$G_t(Q,P) = \frac{1}{c}[F_{t+1}(P,Q) - P_{t+1}F_t(P,Q)] \quad (23)$$

$$F_t(Q,P) = \frac{1}{c}[G_{t+1}(P,Q) - P_{t+1}G_t(P,Q)] \quad (24)$$

the recurrence relations are given by

$$F_{t+1}(P,Q) - (P_{t+1} + Q_t)F_t(P,Q) + (Q_tP_t - c^2I_m)F_{t-1}(P,Q) = 0, \quad (25)$$

$$G_{t+1}(P,Q) - (P_{t+1} + Q_t)G_t(P,Q) + (Q_tP_t - c^2I_m)G_{t-1}(P,Q) = 0. \quad (26)$$

The one dimensional Liapunov exponents for the original system are given by

$$\lambda_{o,i} = \lim_{t\to\infty} \frac{1}{t} \ln \|S_{t,t}\boldsymbol{v}_0\|, \quad (27)$$

where $\boldsymbol{v}_0 \in \mathbf{R}^m$ is the normalized initial value. The one-dimensional Liapunov exponents for the phase coupled system are given by

$$\lambda_{c,i} = \lim_{t\to\infty} \frac{1}{t} \ln \|D^t\boldsymbol{v}_0\|, \quad (28)$$

where $\boldsymbol{w}_0 \in \mathbf{R}^{2m}$ is the normalized initial value. Since

$$F_{t-1}(P,P) + G_{t-1}(P,P) = S_{t,t}, \quad (29)$$

Table 1. Stabilized orbits of a coupled system.

c	u_0	v_0	Period	Exponents	Orbit	Iterations
0.11	−0.88	−0.36	2	−0.0267047	(0.474943,3.52124)	1287
				−0.026728	→ (−9.61302,−1.22096)	
				−1.89661		
				−1.89677		
0.05	0.56	−0.2	4	−0.0611465	(−3.5837,0.756827)	595
				−0.061156	→ (1.14202,2.72359)	
				−1.15308	→ (−11.6142,−1.52877)	
				−2.87185	→ (−0.529614,3.11423)	
0.05	−1	−0.96	8	−0.114072	(−3.67379,0.688937)	414
				−0.493316	→ (1.6563,2.97951)	
				−1.62199	→ (−13.3563,−2.15464)	
				−1.90029	→ (−0.620338,3.25677)	
					→ (−3.89232,0.619765)	
					→ (1.65492,3.01317)	
					→ (−13.2797,−2.13653)	
					→ (−0.647174,3.23894)	
0.045	0.4	0.18	16	−0.0157193	(−13.1894,−2.09875)	2176
				−0.199824	→ (−0.717183,3.20693)	
				−1.83987	→ (−3.2426,0.82956)	
				−2.11901	→ (1.56008,2.88121)	
					→ (−13.1453,−2.07544)	
					→ (−0.667978,3.23201)	
					→ (−3.58586,0.721212)	
					→ (1.62193,2.95178)	
					→ (−13.1811,−2.09528)	
					→ (−0.704024,3.21107)	
					→ (−3.34675,0.797473)	
					→ (1.58441,2.90487)	
					→ (−13.1541,−2.08115)	
					→ (−0.663127,3.22779)	
					→ (−3.63116,0.708811)	
					→ (1.62879,2.95958)	

and using

$$\boldsymbol{w}_0 = \boldsymbol{w}_{01} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \boldsymbol{w}_{02} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{30}$$

with $\boldsymbol{w}_{01}, \boldsymbol{w}_{02} \in \boldsymbol{R}^m$, (27) and (28) become

$$\lambda_{o,i} = \lim_{t\to\infty} \frac{1}{t} \ln \|H_{t-1}(P, P, \boldsymbol{v}_0, \boldsymbol{v}_0)\|, \tag{31}$$

$$\lambda_{c,i} = \lim_{t\to\infty} \frac{1}{t} \ln \left\| \begin{pmatrix} H_{t-1}(P, Q, \boldsymbol{w}_{01}, \boldsymbol{w}_{02}) \\ H_{t-1}(Q, P, \boldsymbol{w}_{02}, \boldsymbol{w}_{01}) \end{pmatrix} \right\|, \tag{32}$$

$$H_t(P, Q, \boldsymbol{v}_1, \boldsymbol{v}_2) = F_t(P, Q)\boldsymbol{v}_1 + G_t(P, Q)\boldsymbol{v}_2. \tag{33}$$

As an application we consider the system [2, 7]:

$$x_{t+1} = a_1 + w_{11}\sigma(x_t) + w_{12}\sigma(y_t), \tag{34}$$

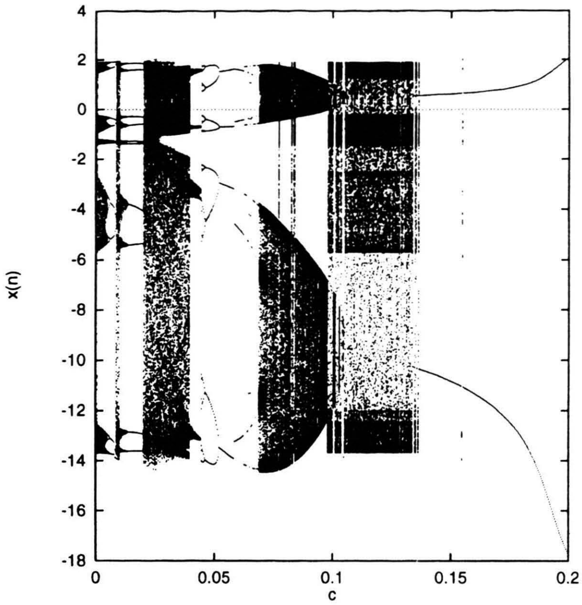

Fig. 1. Projection of bifurcation diagram on c-x plane.

$$y_{t+1} = a_2 + w_{21}\sigma(x_t) + w_{22}\sigma(y_t), \tag{35}$$

$$a_1 = -2, \; w_{11} = -20, \; w_{12} = 6, \tag{36}$$

$$a_2 = 3, \; w_{21} = -6, \; w_{22} = 0, \tag{37}$$

$$\sigma(x) := \frac{1}{1 + e^{-x}}. \tag{38}$$

It models a chaotic neuromodule (recurrent two neuron module with excitatory neuron and an inhibitory neuron with selfconnection) with one-dimensional Liapunov exponents $\lambda_1 = 0.231$ and $\lambda_2 = -3.358$. Stollenwerk and Passeman [2] have found 1 period 1 orbit, 1 period 2 orbit, 1 period 4 orbit, 2 period 5 orbits, 2 period 6 orbits, 3 period 8 orbits, 4 period 9 orbits and 6 period 10 orbits. They examined three control methods for the system namely Ott-Grebogi-Yorke (OGY), least squares and linear least squares methods. Pasemann [7] also studied synchronous and asynchronous chaos in coupled neuromodules.

Our phase coupled system is given by

$$x_{n+1} = a_1 + w_{11}\sigma(x_n) + w_{12}\sigma(y_n) + c(u_n - x_n), \tag{39}$$

$$y_{n+1} = a_2 + w_{21}\sigma(x_n) + w_{22}\sigma(y_n) + c(v_n - y_n), \tag{40}$$

$$u_{n+1} = a_1 + w_{11}\sigma(u_n) + w_{12}\sigma(v_n) + c(x_n - u_n), \tag{41}$$

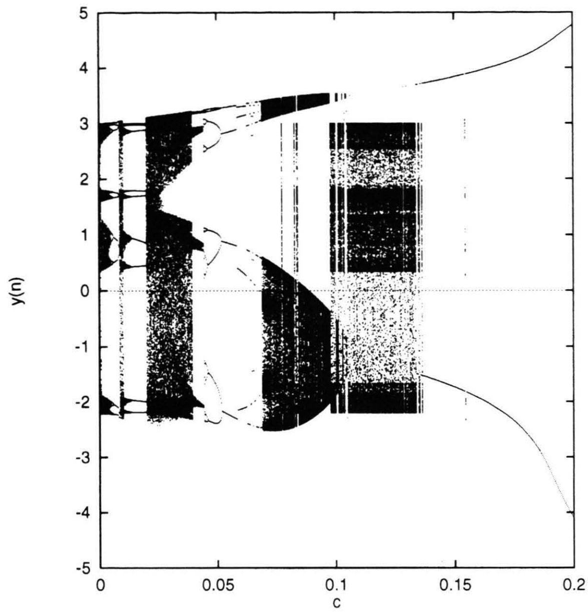

Fig. 2. Projection of bifurcation diagram on c-y plane.

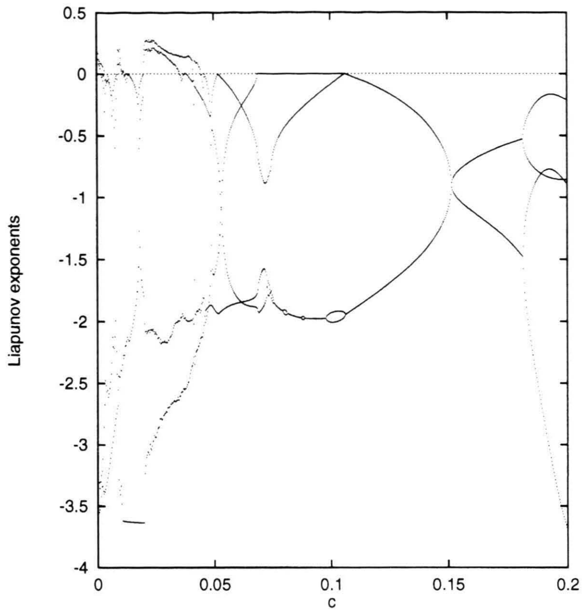

Fig. 3. Liapunov exponents for $c \in [0, 0.2]$.

$$v_{n+1} = a_2 + w_{21}\sigma(u_n) + w_{22}\sigma(v_n) + c(y_n - v_n), \quad (42)$$

where $c \in [0, 1]$ is the coupling constant and $x_0 = 0$, $y_0 = 0$, u_0 and v_0 are the initial conditions. Table 1 lists some of the stabilized orbits, their one dimensional Liapunov exponents and the number of iterations to stabilize (within 20 decimal places). The periodic points reached by phase coupling of chaotic systems are different from the periodic points of the uncoupled system. For $c = 0.05$ we found stabilized periodic orbits for nearly all tested initial values (except 5 of 10000). For the initial values we used

$$u_0, v_0 \in \{-1.0, -0.98, -0.96, \ldots, 0.96, 0.98\}.$$

For the period 2 orbit we find that the average distance from the corresponding points in the period 2 orbit of system (1) is 1.245. For the period 4 orbit we find that the average distance from the corresponding points in the period 4 orbit of system (1) is 0.566. Figures 1 and 2 show the bifurcation diagram projected onto the c-x and c-y planes for $c \in [0, 0.2]$. Figure 3 illustrates the one-dimensional Liapunov exponents for coupling constant $c \in [0, 0.2]$.

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